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Title: *Multilevel Upscaling through Variational Coarsening*

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Multilevel Upscaling through Variational Coarsening

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Abstract.
A new efficient multilevel upscaling procedure for single-phase saturated flow in porous media is presented. While traditional approaches to this problem have focused on the computation of an upscaled hydraulic conductivity, here the coarse-scale model is created explicitly from the fine-scale model through the application of operator-induced variational coarsening. This technique, which originated with robust multigrid solvers, has been shown to accurately capture the influence of fine-scale heterogeneous structure over the complete hierarchy of coarse-scale models that it generates. Moreover, implicit in this hierarchy is the construction of interpolation operators that provide a natural and complete multiscale basis for the fine-scale problem. Thus, this new multilevel upscaling methodology is similar to the Multiscale Finite Element Method (MSFEM) and, indeed, it attains similar accuracy on a variety of problems; yet it is approximately 15 times faster.

1. Introduction

Although the increasing power of modern computer hardware enables effective simulation of many single-scale and single-component systems, there exist fundamental mathematical and algorithmic challenges in the effective simulation of multi-scale and multi-component systems. In particular, a critical underlying problem in the numerical modeling of flow in porous media is the need to resolve the multiscale structure of the subsurface environment. For example, the length scales observed in sedmintary lamine range from the millimeter scale upward, while the simulation domain may be on the order of several kilometers. As a result, fully resolved simulations are computationally intractable, yet the fine-scale variations of the model parameters (e.g., structure and orientation of laminae) significantly affect the properties of the solution at all scales.

Many methods have been proposed to address the complication of fine-scale variation in the material parameters of a porous medium. If certain structural information of the medium is known, then it may be possible to derive a useful coarse-scale model using simple averages. In the case of mean uniform flow, for example, the effective permeability of a medium is bounded between the harmonic and arithmetic averages of the fine-scale permeability, as shown in *Cardwell and Parsons* [1945], along with conditions on the media and flows where these bounds are achieved. The geometric average (*Warren and Price* [1961]) and certain power averages (*Desbarats* [1992a]) provide reliable approximations of the effective permeability if the fine-scale variation in the permeability field satisfies certain conditions. Similarly, in the case of nonuniform flow, *Desbarats* [1992b] demonstrates that a weighted geometric average provides accurate effective transmissivities for low to moderate variances of the log-transmissivity field. However, none of these simple averages accurately capture coarse-scale dynamics of an arbitrary medium under arbitrary flow conditions. In more complex problems, such as transient multiphase flow, coarse-scale models cannot be completely decoupled from the fine-scale dynamics.

The purpose of this paper is to introduce a new multilevel upscaling algorithm, based on variational principles, that accurately and efficiently captures the effects of a multiscale medium. The fine-scale permeability is not explicitly

averaged, neither through a simple average chosen a priori, nor through the solution of local problems as in *Durlofsky* [1991]. In fact, no solution of any fine-scale problems is required by the method. Instead, variational multigrid principles (cf. *Bank et al.* [1985]) are used to construct a self-consistent hierarchy of coarse-scale models directly from the given fine-scale model. Moreover, the solution of a coarse-scale model yields information at more than just that scale, as important finer-scale information is preserved through the hierarchy.

This multilevel approach yields more information than defining equivalent or effective permeabilities over coarse-scale blocks, regardless of whether the coarse-scale permeabilities are calculated using simple averages, as above, or with more sophisticated techniques such as the upscaling of *Durlofsky* [1991] and *He et al.* [2002] that relies on the solution of local fine-scale problems. These techniques, referred to as Laplacian methods in the review by *Wen and Gómez-Hernández* [1996], are more accurate, in general, than simple averages. However, the linear scaling of multigrid iterative solvers for elliptic problems (*Braess and Hackbusch* [1983]) implies that solving the necessary local fine-scale problems is asymptotically no less expensive than solving the global fine-scale problem, unless there is some periodic behavior in the fine-scale permeability that can be exploited. Moreover, such an approach yields only a coarse-scale representation of the flow properties, potentially missing important dynamics necessary to accurately simulate multiphase flows. Comparisons with the Laplacian upscaling of *Durlofsky* [1991] have shown that the multilevel upscaling proposed here is significantly more accurate, as well as computationally more efficient (*MacLachlan* [2004]).

In this paper, we present a comparison with the multiscale finite element method (MSFEM), which was first proposed in *Hou and Wu* [1997] and later analysed by *Hou et al.* [1999]. This method is much closer in both approach and accuracy to the multilevel upscaling ideology considered here. In addition, its application to the simulation of flows in heterogeneous porous media is being actively pursued by the community (*Ye et al.* [2004]). In the MSFEM, multiscale dynamics are captured through the computation of local, low-energy basis functions used to reduce the problem to a chosen coarse scale. The explicit construction of these basis functions accurately represents fine-scale dynamics of the flow; however, this comes at the cost of solution of a set of local fine-scale problems, resulting in an overall cost similar to that of solving the global fine-scale model.

The multilevel upscaling algorithm developed here attains accuracy comparable to the MSFEM approach, but at a significantly lower cost. Both approaches are variational, representing the low-energy components of the model on coarse scales, preserving the minimization properties of the fine-scale finite element discretization. The recursive approach of a multigrid framework, however, allows explicit coarsening of the fine-scale problem without performing any local solves. Instead, physical heuristics are used to identify local characteristics of low-energy error and represent such components on coarser scales.

This paper proceeds as follows. Section 2 introduces the continuum-scale mathematical model of saturated, single-phase flow through porous media and the finite-element discretization that we consider here. In Section 2.1, we discuss the details of variational multigrid, followed by a description of the multilevel upscaling algorithm in Section 2.2. A periodic model problem is discussed in Section 3, where the intuition gained from a simple medium is used to clarify the approach. More complex media are considered in Section 4. Conclusions are presented in Section 5.

2. Background and Method

We consider two-dimensional single-phase saturated flow through a porous medium whose hydraulic conductivity, $\mathcal{K}(\boldsymbol{x})$, is specified on a fine-scale over the domain of interest, Ω. This flow may be modeled at the continuum scale using Darcy's law and conservation of mass,

$$\boldsymbol{q}(\boldsymbol{x}) = -\mathcal{K}(\boldsymbol{x})\nabla h(\boldsymbol{x}), \tag{1}$$

$$\nabla \cdot \boldsymbol{q}(\boldsymbol{x}) = Q(\boldsymbol{x}), \tag{2}$$

for all $\boldsymbol{x} \in \Omega$. Here, $h(\boldsymbol{x})$ is the hydraulic head, $\boldsymbol{q}(\boldsymbol{x})$ is the Darcy flux, and $Q(\boldsymbol{x})$ represents any external sources or sinks of fluid. The conductivity, $\mathcal{K}(\boldsymbol{x})$, is a positive scalar or a positive definite tensor that is assumed to be piecewise smooth with jump discontinuities at interfaces. We consider both no flow boundary conditions (homogeneous Neumann),

$$\boldsymbol{q}(\boldsymbol{x}) \cdot \mathbf{n} = 0\,, \quad \forall \boldsymbol{x} \in \Gamma_{\mathrm{N}}\,, \tag{3}$$

and those of prescribed hydraulic head (Dirichlet).

In the following discussion, we work with this model in its second order form,

$$-\nabla \cdot [\mathcal{K}(\boldsymbol{x})\nabla h] = Q(\boldsymbol{x}) \tag{4}$$

from which a fine-scale discrete model may be obtained with bilinear finite elements on a uniform rectangular mesh that resolves the variation in $\mathcal{K}(\boldsymbol{x})$. Specifically, in the standard Galerkin finite-element formulation, we write

$$h(\boldsymbol{x}) = \sum_{i=1}^{N} h_i \phi_i(\boldsymbol{x}), \tag{5}$$

where $\{\phi_i(\boldsymbol{x})\}$ are the nodal basis functions associated with the rectangular mesh of N nodes. Substitution of (5) into the

weak form of (4) results in a discrete problem that may be expressed as a sparse linear system of equations,

$$A\mathbf{h} = \mathbf{Q}, \tag{6}$$

where $\mathbf{h} = (h_1, \ldots, h_N)^T$ and $\mathbf{Q} = (Q1, \ldots, Q_N)^T$. The elements of the large, sparse $N \times N$ matrix A are given by

$$a_{ji} = \int_\Omega (\mathcal{K}(\boldsymbol{x})\nabla\phi_i(\boldsymbol{x})) \cdot \nabla\phi_j(\boldsymbol{x}) d\Omega. \tag{7}$$

Note that since $\mathcal{K}(\boldsymbol{x})$ is everywhere symmetric and positive-definite, so is A.

2.1. Variational Multigrid Coarsening

A fully resolved mesoscale simulation of flow through strongly heterogeneous media is likely to remain intractable for some time. Thus, an approach is needed to accurately and efficiently capture the influence of fine-scale structure over a hierarchy of coarse-scale models. Many key ingredients of this hierarchy are found in existing multilevel iterative algorithms, such as multigrid. Specifically, these methods achieve their efficiency through the recursive use of successively coarser discrete problems (i.e., a hierarchy of coarse-scale discrete models), in conjunction with a complementary smoothing iteration. In fact, these methods have been shown to scale optimally with N (i.e., solution cost grows only linearly with N) for a broad class of problems, suggesting that scale interaction is well characterized by this approach. Of particular interest here is a class of robust *black box* methods that use the fine-scale discrete model to construct, through a variational principle, the successively coarser coarse-scale operators. Specifically, we use the fundamental components of the *black box multigrid* (BoxMG) algorithm (*Dendy* [1982]) in our multilevel upscaling approach.

An excellent introduction to multigrid methods is given by *Briggs et al.* [2000]. Here, it is sufficient to highlight the key steps in the multigrid solution of Equation (6), which are shown schematically in Figure 1 and described as follows:

- the residual on a particular grid is smoothed (i.e., it must be well approximated on a coarser grid)
- the residual is then *restricted* to the coarser grid
- repeat recursively until the coarsest grid is reached
- solve on the coarsest grid
- *interpolate* a correction to the next finer grid
- smooth the new residual
- repeat to undo the recursive coarsening

We note that the residual of Equation (6), for the j^{th} approximation of the hydraulic head $\mathbf{h}^j$ is simply $\mathbf{r}^j = \mathbf{Q} - A\mathbf{h}^j$.

From this description, it is apparent that the efficiency of a multigrid algorithm is tightly coupled to the effectiveness of the smoother (although this component is beyond the scope of this discussion). Smoothing on coarse levels requires a representation of the fine-scale operator on these levels; we utilize these operators in the upscaling algorithm that follows and, thus, focus here on their specification.

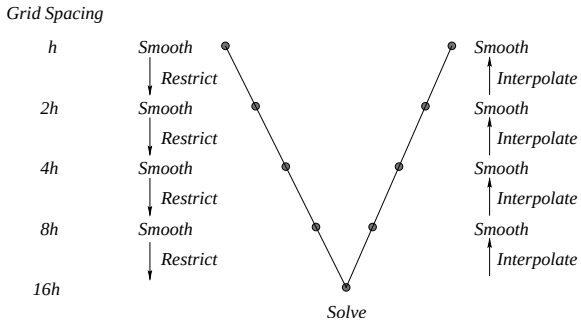

Figure 1. Schematic of a V-cycle multigrid iteration.

First, consider a nested sequence of uniform rectangular meshes labeled $k = 1, \ldots, k_f$, where $k = 1$ denotes the coarsest grid and $k = k_f$ denotes the finest grid. A critical aspect of creating a multigrid algorithm is to define the hierarchy of coarse-grid operators,

$$A_k \quad \text{– discrete operator on grid } k,\ k = 1, 2, \ldots, k_f - 1$$

as well as the intergrid transfer operators,

$$P_{k-1}^k \quad \text{– interpolation operator, } \textit{grid } (k-1) \rightarrow \textit{grid } k$$
$$R_k^{k-1} \quad \text{– restriction operator, } \textit{grid } k \rightarrow \textit{grid } (k-1)$$

Variational coarsening offers one means of defining A_{k-1} in terms of A_k, R_k^{k-1}, and P_{k-1}^k. In particular, restating the discrete linear system as an equivalent minimization problem, and then restricting this minimization to the range of interpolation yields (cf. *Nicolaides* [1979],*Brandt* [1984]),

$$A_{k-1} = \left(P_{k-1}^k\right)^* A_k P_{k-1}^k\,, \tag{8}$$

and, thus, we take $R_k^{k-1} = \left(P_{k-1}^k\right)^*$.

Finally, to complete the specification of a variational coarsening algorithm, we must define the interpolation operator. In fact, the choice of the interpolation operator is critical to the robustness and efficiency of the resulting multigrid algorithm. For example, a naive choice such as bilinear interpolation erroneously assumes that the gradient of the hydraulic head is continuous and generates a coarse-scale model in which the upscaled hydraulic conductivity is simply an arithmetic average of the fine-scale conductivity. Thus, it is not surprising that bilinear interpolation leads to a fragile multigrid algorithm that is not suitable for

strongly heterogeneous media. Instead, a significantly better approach is to use entries in the fine-scale discrete operator to define an interpolation that preserves certain fundamental properties of the solution. This technique was dubbed operator-induced interpolation by *Dendy* [1982] and was shown by *Moulton et al.* [1998] to approximately enforce the continuity of the normal component of the Darcy flux across interfaces.

2.2. The Multilevel Upscaling (MLUPS) algorithm

The variational coarsening presented in Section 2.1 generates a complete hierarchy of coarse-scale models, A_k, with very useful properties. For example, it preserves the symmetry and definiteness of the fine-scale operator, A_{k_f}, and, at each level, minimizes the error in the range of the interpolation (cf. *Nicolaides* [1979], *Brandt* [1984]). More importantly for upscaling applications, it implicitly generates multiscale basis functions (*Grauschopf et al.* [1997]). To clarify this property, denote, on each scale k, the set of basis functions $\{\psi_j^k\}$, which we define recursively from the finest scale. On the scale of discretization, grid k_f, the basis functions are simply the bilinear basis functions used in Equation (5), $\psi_j^{k_f} = \phi_j$, for all fine-scale nodes j. Given an operator, A_k, on level k, generated by basis functions ψ_j^k,

$$(A_k)_{ij} = \int_\Omega \langle \mathcal{K}(\boldsymbol{x}) \nabla \psi_j^k, \nabla \psi_i^k \rangle \, d\Omega, \tag{9}$$

denote the elements of the interpolation as $P_{k-1}^k = p_{ij}$, such that substitution in (8) gives

$$\begin{aligned} \left(A_{k-1}\right)_{ij} &= \sum_{l,m} p_{li} p_{mj} \int_\Omega \langle K(x) \nabla \psi_m^k, \nabla \psi_l^k \rangle d\Omega \qquad (10) \\ &= \int_\Omega \left\langle K(x) \nabla \Big(\sum_m p_{mj} \psi_m^k \Big), \nabla \Big(\sum_l p_{li} \psi_l^k \Big) \right\rangle d\Omega. \end{aligned}$$

Hence, if we define the new multiscale basis functions on level $k-1$ as

$$\psi_j^{k-1} = \sum_m p_{mj} \psi_m^k \,, \tag{11}$$

we may write the discrete coarse-grid operator in the form

$$\left(A_{k-1}\right)_{ij} = \int_\Omega \Big(\mathcal{K}(\mathbf{x}) \nabla \psi_i^{k-1}, \nabla \psi_j^{k-1} \Big) d\Omega \,. \tag{12}$$

Therefore, interpolation provides not only mappings between grid function spaces, but may also be viewed as part of the discretization on coarse grids. It is in this way that the variational definition of the coarse-grid operator may be viewed as a discrete method for generating a hierarchy of coarse-scale discrete models that accurately capture the influence of the fine-scale heterogeneous structure.

Our new multilevel upscaling (MLUPS) algorithm uses the components of the black box multigrid algorithm in the following way:

1. Create a conforming bilinear discretization of (4) on the finest grid, assuming homogeneous Neumann boundary conditions.
2. Perform operator-induced variational coarsening to construct the interpolation operators, P_{k-1}^k, and the hierarchy of coarse-scale operators, A_k, according to Equation (8)
3. Select a coarse-scale, $k = k_c$, on which to solve the coarse-scale approximation of Equation (6).
4. Restrict the fine-scale boundary conditions and the fine-scale source to level k_c, then use black box multigrid to solve this coarse-scale problem.
5. Interpolate the solution to the finest grid using the multiscale basis functions defined by Equation (11).

3. Periodic Media

We first apply the MLUPS algorithm (Section 2.2) to a model problem with a two-scale periodic variation in the hydraulic conductivity. Specifically, we consider a structured pattern of square inclusions of a high conductivity medium against a homogeneous background, as depicted on the left of Figure 2. The conductivity is constructed by a four by four tiling of the unit cell shown on the right of Figure 2, where the $\mathcal{K}(\boldsymbol{x}) = 1000$ inside the dark region and $\mathcal{K}(\boldsymbol{x}) = 1$ in the background medium. This regular pattern provides an ideal setting to develop intuition into the MLUPS approach as the resulting multiscale basis functions clearly display the influence of this structure on the flow. Results for randomly generated anisotropic heterogeneous media are presented in the following section.

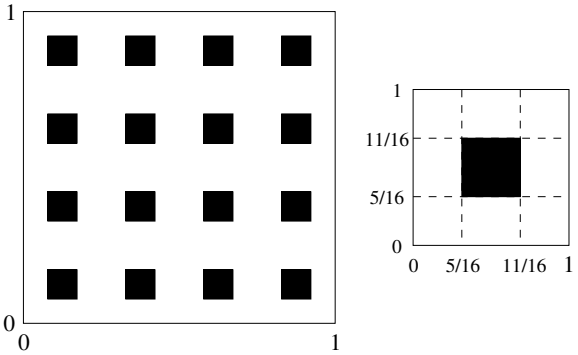

Figure 2. Periodic conductivity field.

One approach to visualizing the MLUPS multiscale basis functions is to associate each basis function with a coarse-grid node on each level of the multigrid hierarchy. In this case, as we traverse the hierarchy to coarser scales, the support of a basis function grows and, hence, its shape captures an increasing region of fine-scale structure. In our current work with BoxMG, we use standard coarsening, which takes half of the points in each coordinate direction at each level. Thus, the number of points is reduced by a factor of four on each level and, moreover, the support of the multiscale basis function grows in area by a factor of four with each level.

However, this approach is not particularly informative for this model problem because it takes too many levels for the support of the coarse-scale basis functions to cover interesting features of the conductivity field. Instead, we examine how the information in the multigrid interpolation operators compounds from the coarsest scales up to the finest. Figure 3 shows this development on the coarse scales, where we see the evolution of the basis function centered at $(\frac{1}{2}, \frac{1}{2})$ from the bilinear basis function normally used on any scale to the effective MLUPS basis function for these scales in Figure 4.

The MLUPS basis function exhibits many features appropriate for this flow. In the regions where $\mathcal{K}(\boldsymbol{x})$ is large, we expect small gradients of head relative to those in the background medium (so that $\mathcal{K}\nabla h$ is relatively constant and $\nabla \cdot \mathcal{K}\nabla h = 0$); the basis function reflects this with plateaus in its contours. Outside these regions, we do not know, a priori, what to expect of the hydraulic head for a general flow situation, and so bilinear tendencies are retained to best fit general heads on the coarse scale.

To demonstrate the accuracy of the MLUPS basis functions, we consider Equation (4), subject to no-flow boundary conditions on the top and bottom edges, with heads $h = 1$ along $x = 0$ and $h = 0$ along $x = 1$. We then compute errors in both the hydraulic head and the average normal flux relative to the fine-scale *true* solution obtained on a 256×256 with standard bilinear finite elements. Specifically, the errors in the head are measured in the discrete vector norms approximating $L_2(\Omega)$,

$$\|e(h)\|_2 = \left(\frac{1}{N} \sum_{i=1}^{N} e(h)_i \right)^{\frac{1}{2}}, \tag{13}$$

where N is the number of nodes on the fine mesh, and $L_\infty(\Omega)$,

$$\|e(h)\|_\infty = \max_i |e(h)_i|. \tag{14}$$

Similarly, to quantify the accuracy of the Darcy flux we consider the average flux through the domain,

$$\boldsymbol{q}_x = \int_0^1 (\boldsymbol{q} \cdot \hat{\boldsymbol{x}}) dy = \int_0^1 \left[-\mathcal{K}(x,y) \nabla h(x,y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] dy, \tag{15}$$

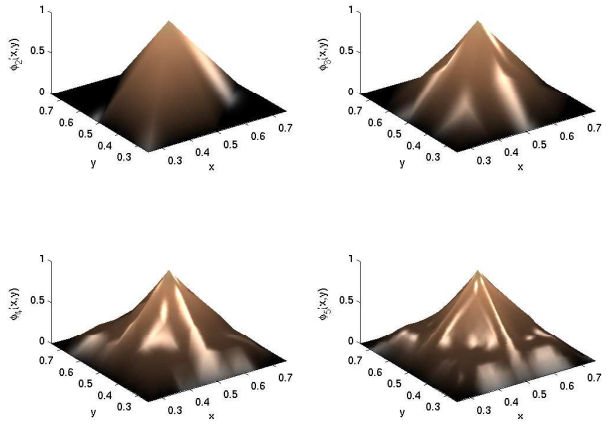

Figure 3. Coarse-scale basis functions for the periodic problem.

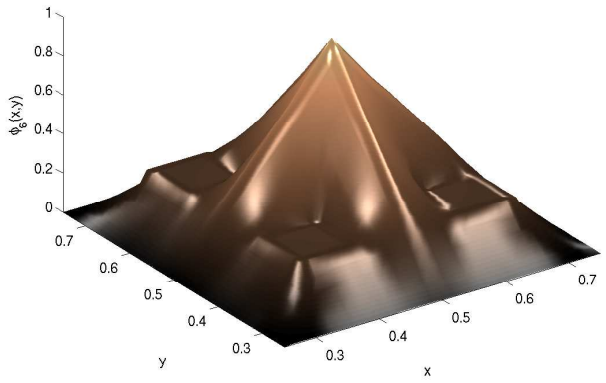

Figure 4. Fine-scale basis function for periodic problem

and define the corresponding discrete vector norms approximating $L_2([0,1])$,

$$\|e(\boldsymbol{q}_x)\|_2 = \left(\frac{1}{N_x} \sum_{i=1}^{N_x} e(\boldsymbol{q}_x)_i^2 \right)^{\frac{1}{2}}, \tag{16}$$

where N_x is the number of nodes in the x-direction on the fine mesh, and $L_\infty([0,1])$,

$$\|e(\boldsymbol{q}_x)\|_\infty = \max_i |e(\boldsymbol{q}_x)_i|. \tag{17}$$

These results are summarized in Table 1, where the computational fine-scale ranges from 64×64 elements (a standard bilinear discretization) to 8×8 MLUPS elements. Note that as the number of degrees of freedom on the computational scale decreases, we see increasing L_2 errors in both head and average flux. The L_∞ error in pressure, however, remains relatively constant. This max-norm error is attained along

Table 1. Errors in flow properties for periodic conductivity

Grid	$\|e(p)\|_2$	$\|e(p)\|_\infty$	$\|e(\boldsymbol{q}_x)\|_2$
64^2	4.29×10^{-4}	4.54×10^{-3}	7.45×10^{-3}
32^2	5.52×10^{-4}	3.92×10^{-3}	2.29×10^{-2}
16^2	1.04×10^{-3}	4.02×10^{-3}	7.16×10^{-2}
8^2	1.54×10^{-3}	6.62×10^{-3}	3.67×10^{-2}

lines of constant y, midway between the high-conductivity inclusions, where the MLUPS method tends to undershoot the exact solution producing a small cusp in the y-direction instead of the smooth but rapidly varying profile of the fine-scale conductivity.

To better understand this performance, recall that we discretized Equation (4) with finite elements, and, specifically, that the solution of the computational fine-scale equations is the same as that of the minimization problem,

$$h = \underset{\tilde{h} \in \mathcal{V}}{\operatorname{argmin}} \iint \left(\frac{1}{2} (\mathcal{K} \nabla \tilde{h}) \cdot \nabla \tilde{h} - Q \tilde{h} \right) d\boldsymbol{x}.$$

The MLUPS method may be viewed as picking a subset of the space, $\mathcal{V}$, over which to perform the minimization. The natural weighting in the minimization functional, however, emphasizes errors in the regions where $\mathcal{K}(\boldsymbol{x})$ is large and/or the gradient of the hydraulic head is large. In a sense, this forces MLUPS to choose basis functions that allow small errors in the head in the background medium so that it can match the head more accurately in the high conductivity regions. Such an approach is consistent with the fine-scale finite element discretization and the variational framework of the Galerkin Finite Element Method and is thus viewed as a feature of the method.

4. Anisotropic Heterogeneous Media

While the example in Section 3 is useful as an introduction to the MLUPS methodology, it does not represent typical conductivity fields for which coarse-scale models are needed. A more realistic representation may be obtained with a random field generator. Thus, in this section, we focus on realizations of heterogeneous layered media, and examine the effect of such layering on the performance of the MLUPS method. Specifically, we consider a series of problems in which the hydraulic conductivity, $\mathcal{K}(\boldsymbol{x})$, is generated using the GSLIB software package (*Deutsch and Journel* [1998]). The computational fine scale is selected as a 256×256 element grid, and on each element the hydraulic conductivity is taken to be a constant. A principle axis of statistical anisotropy is selected between 0 and 90 degrees, relative to the positive x-axis, and a conductivity field is generated such that $\log_{10}(\mathcal{K}(\boldsymbol{x}))$ is normally distributed with mean zero and variance 4, with correlation lengths of 0.8 along the principle axis and 0.04 in the direction orthogonal to this axis.

Samples of these conductivity fields are shown in Figure 5, for the angles to the principle axis of 30° on the left and 45° on the right. The grayscale in these figures represents a range in conductivity from approximately 10^{-3} (white) to 10^3 (black). Notice the strong layering that results from the significant difference in the correlation lengths along and orthogonal to the chosen axis. The MLUPS basis functions for the node at $(\frac{1}{2}, \frac{1}{2})$ after upscaling by a factor of 32 in each direction are shown in Figure 6 for 30°, on the left, and 45°, on the right. These basis functions strongly reflect the fine-scale features of the conductivity. Most noticeable is the rotation in the features to match the angles of the layering in the conductivity, visible along the lower-right edges of the basis functions. Notice also how these basis functions strongly represent the expected behavior of the head for general flow conditions, with relatively small gradients in regions of large conductivity and relatively large changes in regions where the conductivity values are small.

Because of the similarity in approach of the two methods, we compare the results generated by the MLUPS method with the MSFEM of *Hou and Wu* [1997], and *Hou et al.* [1999]. The MSFEM method considers a given fine computational scale and explicitly creates basis functions that vary on that scale to use in the coarse-scale discretization. These functions are constructed by solving local fine-scale problems with boundary conditions especially chosen to form coarse-scale nodal basis functions. In this study, we used the oscillatory boundary conditions advocated in [*Hou and Wu*, 1997, Section 2.2], as our tests with the alternative (and more expensive) technique of oversampling did not yield any gain in accuracy.

Once again we consider measures of error in both the head and average normal flux for uniform mean-flow from left to right across the domain. No-flow boundary conditions are applied at the top and bottom, while hydraulic head is prescribed on the left and right edges, with $h(0, y) = 1$ and $h(1, y) = 0$. The computational fine-scale discretization on the 256×256 element mesh uses the standard bilinear Galerkin finite element discretization (BLFEM) outlined in Section 2. An overresolved calculation on a 2048×2048 element mesh is used to represent the true solution of this problem, and the BLFEM, MLUPS, and MSFEM solutions are represented at the nodes of this mesh through their basis function representations. Errors in both the hydraulic head and the average flux are calculated relative to the fine-scale solution, measured using the discrete vector norms defined in Equations (13) through (17).

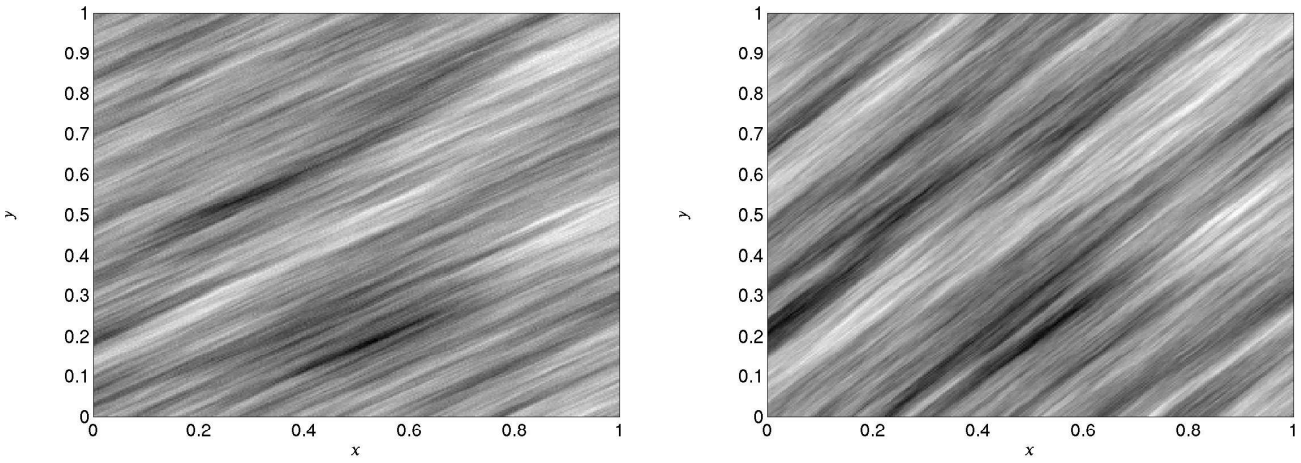

Figure 5. Conductivity field for 30° axis of anisotropy (left) and 45° (right).

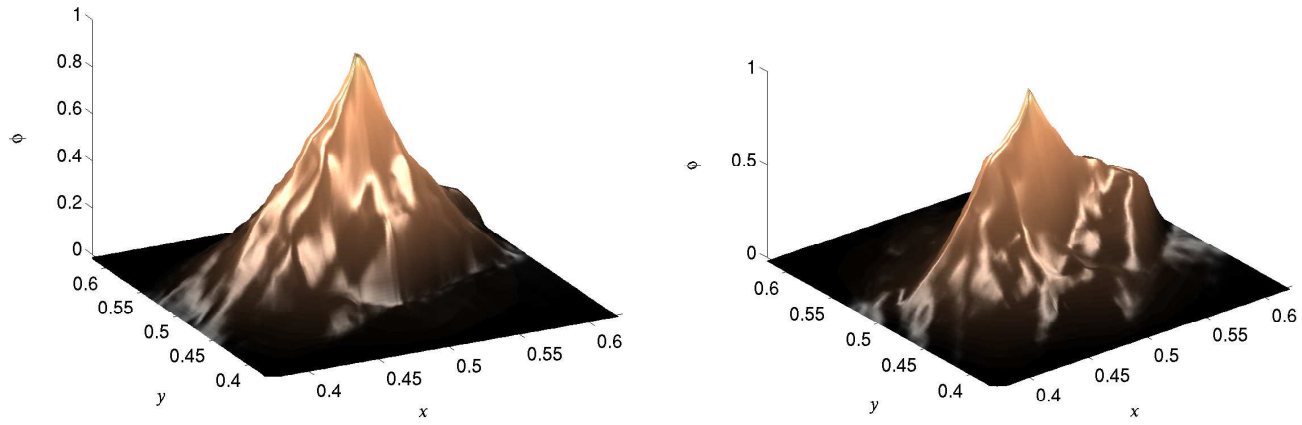

Figure 6. MLUPS basis function for the node $(\frac{1}{2}, \frac{1}{2})$ on an 8×8 grid for 30° axis of anisotropy (left) and 45° (right).

Plots of the hydraulic head errors for the BLFEM, MLUPS, and MSFEM computations on 5 degree steps of the orientation of the conductivity layers are shown in Figure 7, with the $L_2(\Omega)$ given by Equation (13) on the left, and the $L_\infty(\Omega)$ given by Equation (14) on the right. The errors for the bilinear solution on the 256×256 element grid (BLFEM) are presented here as a baseline for the upscaling methods. In some sense, these represent the best we can expect in the upscaled results, illustrating the error in the solution of the computational fine-scale model. In these plots, we see that the MLUPS and MSFEM methods produce solutions with errors of similar magnitude for these test problems. The MLUPS technique appears to be slightly more accurate for small angles in the orientation of the conductivity layers, although such a generalization would not be possible without large ensemble averages that were not part of this study. For larger angles, the errors of the two methods are comparable, with moderate oscillation in the magnitude of the errors for both techniques.

Plots of the errors in average normal flux, integrated along lines of constant x, are shown in Figure 8. The $L_2([0,1])$ norm of average normal flux is computed according to Equation (16) and is shown on the left, while the $L_\infty([0,1])$ is given by Equation (17) and is shown on the right. Once again we note that the overall similarity in accuracy between the two methods in these error measures, although we see relatively large excursions in computed fluxes for the MSFEM solution for small angles. In general, significant fluctuation with x in the computed average fluxes was seen with the MSFEM procedure, whereas the MLUPS computations were more consistent. This can be seen in the closer relationship between the L_2 and L_∞ norms of the flux for the MLUPS method.

Given the similar accuracy of these two methods, the

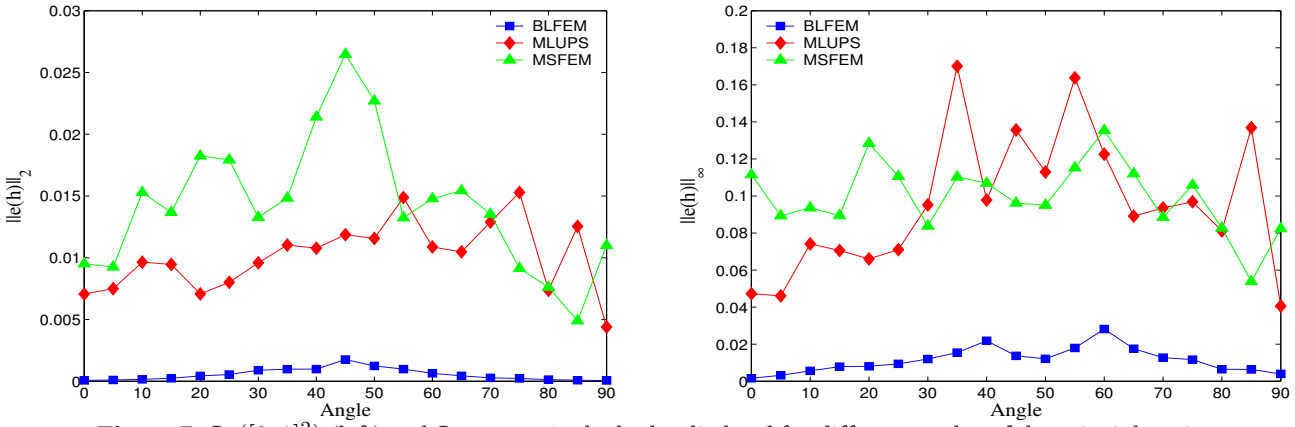

Figure 7. $L_2([0,1]^2)$ (left) and L_∞ errors in the hydraulic head for different angles of the principle axis.

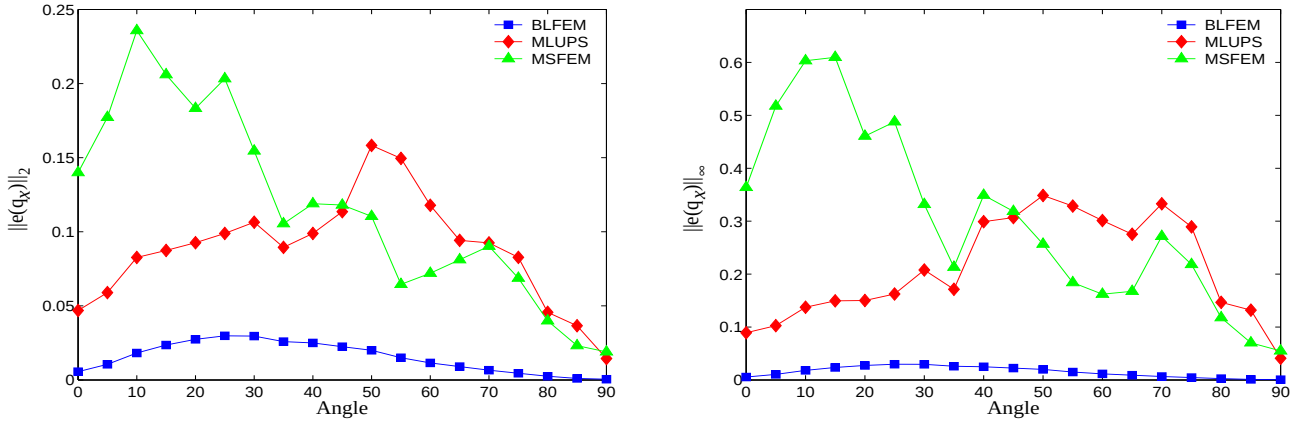

Figure 8. $L_2([0,1])$ error (left) and L_∞ error (right) in the average normal flux for different angles of the principle axis.

MLUPS methodology is much more attractive due to its lower computational cost. Specifically, the MSFEM computation requires three fine-scale solves per coarse-scale element to create a full set of basis functions (*Hou and Wu* [1997]). With the existence of optimal solvers such as multigrid for this class of problems, this setup phase has a cost that is proportional to the cost of solving the computational fine-scale system. In fact, using BoxMG (*Dendy* [1982]) to solve the 256×256 element computational fine-scale problems, solution time ranges from approximately 1 to 2.5 seconds on a system with a 1.6 GHz Athlon processor. This variation in solution time is due to the variations in the realizations of the conductivity across different angles, resulting in a degradation in the smoothing rate for some configurations, and hence, ultimately a degradation in the convergence of BoxMG. The corresponding MSFEM calculations for upscaling to an 8×8 coarse computational-scale mesh consistently required 1.8 seconds of CPU time on the same machine. In contrast, the MLUPS calculation consistently needed only 0.12 seconds of CPU time, one-fifteenth of the MSFEM cost. Because the coarse element-scale computation for MSFEM, and the computational coarse grids were so small, the MSFEM and MLUPS approaches did not see the same fluctuation in CPU time as the fine-scale computations did.

5. Conclusions

We demonstrate the capability of the multilevel upscaling methodology to efficiently generate a complete hierarchy of self-consistent coarse-scale models, as well as the corresponding multiscale basis functions, thereby facilitating the computation of coarse- and fine-scale properties of the solution. In this study, our multilevel upscaling algorithm demonstrated accuracy comparable to the popular Multiscale Finite Element Method but at significantly lower computational cost. Moreover, it provides a natural setting for adaptivity, error estimation, and extensions to more complex regimes such as unsaturated, multiphase, and reactive flows.

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